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Abstract
There has recently been interest in relating properties of matrices drawn at
random from the classical compact groups to statistical characteristics of
number-theoretical *L*-functions. One example is the relationship conjectured
to hold between the value distributions of the characteristic polynomials of
such matrices and value distributions within families of *L*-functions. These
connections are extended here to non-classical groups. We focus on an explicit
example: the exceptional Lie group G_2. The value distributions for character-
istic polynomials associated with the 7- and 14-dimensional representations of
G_2, defined with respect to the uniform invariant (Haar) measure, are calculated
using two of the Macdonald constant term identities. A one-parameter family
of *L*-functions over a finite field is described whose value distribution in the
limit as the size of the finite field grows is related to that of the characteristic
polynomials associated with the seven-dimensional representation of G_2. The
random matrix calculations extend to all exceptional Lie groups.

PACS numbers: 02.10.Yn, 02.10.De, 02.20.−a

1. Introduction

Most work on the connection between random matrix theory and *L*-functions has concentrated
on random matrices chosen from ensembles related to the *classical* compact groups.
Montgomery [19], Rudnick and Sarnak [23] and Bogomolny and Keating [1, 2] calculated
the correlation functions of the zeros of the Riemann zeta-function, scaled to have unit mean
spacing, in the limit as *T*, the extent of the averaging range up the critical line, tends to
infinity. Their results suggest that these correlation functions coincide with those relating to
the eigenvalues of unitary matrices in the limit as the matrix size, *N*, tends to infinity. In the
latter case the average is defined with respect to the uniform invariant (Haar) measure on the
unitary group $U(N)$, that is, with respect to the circular unitary ensemble (CUE) of random
matrix theory (RMT). There is extensive numerical evidence in support of this connection

[20], which is expected to extend to the zeros of any given principal *L*-function. Katz and Sarnak [15] conjectured that the distributions of low-lying zeros in families of *L*-functions are the same as those of the eigenvalues of matrices from the various classical compact groups (e.g., the orthogonal group $O(N)$ and the symplectic group $USp(2N)$, as well as $U(N)$), the particular group in question being determined by the symmetry of the family. This is also supported by numerical evidence [22].

It was suggested by Keating and Snaith [16] that the value distribution of a given principal *L*-function on its critical line coincides, in the limit as $T \to \infty$, with the value distribution of the characteristic polynomials of random unitary matrices, defined again by an average with respect to Haar measure for $U(N)$, in the limit as $N \to \infty$. The random matrix value distribution was calculated in [16] by expressing the group average in terms of an integral over the matrix eigenvalues, using a formula for the measure due to Weyl [28], and then relating the resulting (*N*-dimensional) integral to the one evaluated by Selberg. This idea was later extended in line with the Katz–Sarnak philosophy to relate the value distribution within a given family of *L*-functions at the centre of the critical strip to the value distribution of the characteristic polynomials associated with elements of the appropriate classical compact group in the $N \to \infty$ limit [17, 3, 4]. Again, the random matrix calculations were performed using Weyl's integration formula and the Selberg integral. One interesting feature of these calculations is that in all cases the logarithm of the characteristic polynomial, normalized appropriately (by $\log N$), satisfies a central limit theorem in the limit $N \to \infty$. This is in agreement with a theorem of Selberg which states that the logarithm of the Riemann zeta-function, normalized appropriately (by $\log \log T$), also satisfies a central limit theorem in the limit $T \to \infty$. For further related developments see [10, 11, 5].

It is in this context that we now ask whether there is a connection between *L*-functions and random matrices from the *non-classical* groups. A particularly interesting class of these groups, closely related to the classical groups, is that of the exceptional Lie groups. Our purpose in this paper is to point out that a number of key constructions which serve to provide the link with random matrix theory in the classical case have analogues for the exceptional Lie groups. We illustrate this by computing the moments and value distribution of the characteristic polynomials of matrices associated with the 7- and 14-dimensional representations of one particular exceptional Lie group, G_2. The methods employed are again the appropriate Weyl integration formula and generalizations of the Selberg integral conjectured by Macdonald (and known as Macdonald constant term identities) [18], proved for G_2 by Zeilberger [29] and Habsieger [9] and by Opdam [21] in the general case. These methods extend to all of the other exceptional Lie groups. We then describe a one-parameter family of *L*-functions over a function field, whose value distribution coincides with that of the characteristic polynomials associated with the seven-dimensional representation of G_2 in the limit as the size of the finite field grows (this was proved by Katz [14]). The link with finite fields is natural, because *N* is fixed for the exceptional groups and the *L*-functions in question (whose zeros correspond to eigenvalues) are polynomials.

This paper is organized as follows. In section 2 we review properties of G_2 necessary for our random matrix calculations. These calculations are performed in section 3. The *L*-functions associated with G_2 are constructed in section 4. In section 5 we conclude with a brief discussion of the generalization to the other exceptional Lie groups.

2. Preliminaries about G_2

As pointed out in the introduction, the exceptional Lie groups are closely related to the classical matrix groups. One particularly natural way of seeing this relationship is via their Lie algebras.

An important class of Lie algebras, because they form building blocks of more general algebras, is the class of complex semi-simple Lie algebras. This class allows a complete categorization and is elegantly summarized in the possible Dynkin diagrams which encapsulate the allowed root systems; the root systems describe the structure constants of the Lie algebra (standard references for this material include [7, 8, 6]). The result of the analysis is that the structure of possible root systems is highly constrained. Indeed the only possibilities fall into four infinite families, a_n, b_n, c_n and d_n, plus five exceptional cases, g_2, f_4, e_6, e_7 and e_8. Each complex Lie algebra has a compact real form and this real form is the Lie algebra of a compact group; the compact real forms of a_n, b_n, c_n and d_n are the Lie algebras of $SU(n+1)$, $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$, respectively. The five exceptional cases are the Lie algebras of the groups G_2, F_4, E_6, E_7 and E_8. What is significant for our purposes is that many of the constructions which exist for the classical groups have analogues for the exceptional groups.

Here, for concreteness, we will mainly focus on the smallest exceptional case, namely G_2, for which the Lie algebra is the compact real form of g_2. The group G_2 is the automorphism group of the octonions, and has an embedding into $SO(7)$.

The group is 14-dimensional and has rank 2. The six positive roots may be taken to be

$$
\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \alpha_2 = \begin{pmatrix} -3/2 \\ \sqrt{3}/2 \end{pmatrix}
$$

$$
\alpha_3 = \alpha_1 + \alpha_2 = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \qquad \alpha_4 = 2\alpha_1 + \alpha_2 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \tag{1}
$$

$$
\alpha_5 = 3\alpha_1 + \alpha_2 = \begin{pmatrix} 3/2 \\ \sqrt{3}/2 \end{pmatrix} \qquad \alpha_6 = 3\alpha_1 + 2\alpha_2 = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}.
$$

The complete set of roots is $R = \{\pm\alpha_i\}$, $i = 1, \ldots, 6$. The set of short roots is $R_S = \{\pm\alpha_1, \pm\alpha_3, \pm\alpha_4\}$; the set of long roots is $R_L = \{\pm\alpha_2, \pm\alpha_5, \pm\alpha_6\}$. The Weyl group W of G_2 is a dihedral group with 12 elements.

Let T be a maximal torus of G_2, which is isomorphic to a product of two circles $S^1 \times S^1$. Every element of G_2 is conjugate in G_2 to an element of T, which is unique up to conjugation by the Weyl group.

Weyl's integration formula reads: if $d\mu_{\text{inv}}(g)$ is the Haar probability measure on G_2, dt is the Haar probability measure on T and F is a continuous function on G_2, invariant under conjugation, then

$$
\int_{G_2} F(g)\, d\mu_{\text{inv}}(g) = \frac{1}{12} \int_T F(t)|\Delta(t)|^2\, dt \tag{2}
$$

where

$$
\Delta(t) = \sum_{\sigma \in W} (\det \sigma) t^{\sigma(\delta)} = t^\delta \prod_{\alpha > 0} (1 - t^{-\alpha}) \tag{3}
$$

(the equality is Weyl's denominator formula), where δ is half the sum of the positive roots:

$$
\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha = 5\alpha_1 + 3\alpha_2. \tag{4}
$$

and $\sigma(\delta)$ means the vector obtained from δ by the Weyl group element σ. If we parametrize a particular maximal torus by $t = (t_1, t_2)$, then t^α is an expression of the form $t_1^{\alpha.e_1} t_2^{\alpha.e_2}$ where e_1 and e_2 are (two-component) vectors; the vectors e_1 and e_2 and range of the parameters t_1 and t_2 depend on the choice of maximal torus. We derive our results below without needing to make an explicit choice for this torus.

For any pair of integers $[n_1, n_2]$ there is an irreducible representation $\rho_{[n_1, n_2]}$ which has highest weight $\lambda_{[n_1, n_2]} = n_1 \omega_1 + n_2 \omega_2$, where $\omega_1 = \alpha_4$ and $\omega_2 = \alpha_6$ are the fundamental weights.

The character χ_λ of the representation evaluated at the group element $t \in T$ is

$$\chi_\lambda(t) = \text{Tr}[\rho_\lambda(t)] = \sum_\mu d_\mu t^\mu \tag{5}$$

where μ are the weights of the representation and d_μ is the multiplicity of the weight μ.

The character $\chi_\lambda(t)$ and dimension d_λ of the representation ρ_λ are given by Weyl's formulae:

$$\chi_\lambda(t) = \frac{\sum_{\sigma \in W} (\det \sigma) t^{\sigma(\lambda + \delta)}}{\sum_{\sigma \in W} (\det \sigma) t^{\sigma(\delta)}} \tag{6}$$

$$d_\lambda = \frac{\prod_{\alpha > 0} (\lambda + \delta).\alpha}{\prod_{\alpha > 0} \delta.\alpha}. \tag{7}$$

The sums in (6) are over elements σ in the Weyl group. With these conventions, the orthogonality relation for characters is

$$\frac{1}{12} \int_T |\Delta(t)|^2 \chi_{[n_1, n_2]}(t) \overline{\chi_{[m_1, m_2]}(t)} \, dt = \delta_{n_1, m_1} \delta_{n_2, m_2}. \tag{8}$$

We will be particularly interested in the fundamental representations $[1, 0]$ (induced from the embedding of G_2 into $SO(7)$) and $[0, 1]$ (the adjoint representation), which have characters

$$\chi_{[1,0]}(t) = 1 + \sum_{\alpha \in R_S} t^\alpha \tag{9}$$

$$\chi_{[0,1]}(t) = 2 + \sum_{\alpha \in R} t^\alpha \tag{10}$$

and dimensions

$$d[1, 0] = 7 \qquad d[0, 1] = 14. \tag{11}$$

3. Characteristic polynomials

We will focus on the characteristic polynomials

$$Z(U_\rho, \theta) := \det(1 - U_\rho \, e^{-i\theta}) \tag{12}$$

of (unitary) matrices U_ρ coming from a given representation ρ of the group. The group elements which these matrices represent can be thought of as being chosen randomly from the group with respect to the uniform invariant (Haar) measure. We will calculate explicit expressions for the averages (over the group, with respect to Haar measure) of $|Z|^s$ for complex numbers s (see [16, 17] for analogous calculation relating to $U(N)$, $O(N)$ and $USp(2N)$).

Let us first consider the modulus of Z. We wish to calculate

$$\langle |Z(U_\rho, \theta)|^s \rangle_{G_2} = \int |Z(U_\rho(g), \theta)|^s \, d\mu_{\text{inv}}(g)$$

$$= \int |\det(1 - U_\rho(g) \, e^{-i\theta})|^s \, d\mu_{\text{inv}}(g). \tag{13}$$

Since the integrand is a class function, this integral reduces to an integral over the maximal torus T:

$$\langle |Z(U_\rho, \theta)|^s \rangle_{G_2} = \frac{1}{12} \int_T |\Delta(t)|^2 |\det(1 - U_\rho(t) \, e^{-i\theta})|^s \, dt. \tag{14}$$

3.1. The seven-dimensional representation

This representation is induced by the embedding of G_2 as a subgroup of $SO(7)$. From (9), we can calculate that

$$Z(U_{[1,0]}, \theta) = \det(1 - U_{[1,0]}(t)\, e^{-i\theta}) = (1 - e^{-i\theta}) \prod_{\alpha \in R_S} (1 - t^\alpha\, e^{-i\theta}). \qquad (15)$$

We see that Z has a zero at $\theta = 0$, as is the case for the characteristic polynomial of any odd-dimensional orthogonal matrix. Let us define $\hat{Z}$ as

$$\hat{Z}(U_{[1,0]}, \theta) = (1 - e^{-i\theta})^{-1} Z(U_{[1,0]}, \theta). \qquad (16)$$

We now present a formula for $\langle |\hat{Z}(U_{[1,0]}, \theta)|^s \rangle$ at $\theta = 0$.

We have that

$$\langle |\hat{Z}(U_{[1,0]}, 0)|^s \rangle_{G_2} = \frac{1}{12} \int_T |\Delta(t)|^2 | \prod_{\alpha \in R_S} (1 - t^\alpha)|^s \, dt$$

$$= \frac{1}{12} \int_T |\Delta(t)|^2 \prod_{\alpha \in R_S} (1 - t^\alpha)^s \, dt \qquad (17)$$

but

$$|\Delta(t)|^2 = \prod_{\alpha \in R} (1 - t^\alpha) \qquad (18)$$

and so

$$\langle |\hat{Z}(U_{[1,0]}, 0)|^s \rangle_{G_2} = \frac{1}{12} \int_T \prod_{\alpha \in R} (1 - t^\alpha)^{k_\alpha} \, dt \qquad (19)$$

where

$$k_\alpha = \begin{cases} s+1 & \text{if} \quad \alpha \in R_S \\ 1 & \text{if} \quad \alpha \in R_L. \end{cases} \qquad (20)$$

Consider first the case that s is an integer. The value of the integral (19) is then the constant term in the expression

$$\frac{1}{12} \prod_{\alpha \in R} (1 - t^\alpha)^{k_\alpha}. \qquad (21)$$

The value of this constant term is in turn one of Macdonald's celebrated constant term conjectures [18], proved for G_2 by Zeilberger [29] and Habsieger [9] (see Opdam [21] for a uniform proof):

$$\frac{(3k_S + 3k_L)!(2k_S)!(2k_L)!(3k_L)!}{12(2k_S + 3k_L)!(k_S + 2k_L)!(k_S + k_L)!(k_S)!((k_L)!)^2} \qquad (22)$$

where k_S (resp. k_L) is the value of k_α for the short (resp. long) roots.

Thus for the representation $[1, 0]$, $k_S = s + 1$ and $k_L = 1$, and so for s a positive integer or zero,

$$\langle |\hat{Z}(U_{[1,0]}, 0)|^s \rangle_{G_2} = \frac{(3s + 6)!(2s + 2)!}{(2s + 5)!(s + 3)!(s + 2)!(s + 1)!}. \qquad (23)$$

It follows from Carlson's theorem (see [27]) that then

$$\langle |\hat{Z}(U_{[1,0]}, 0)|^s \rangle_{G_2} = \frac{\Gamma(3s + 7)\Gamma(2s + 3)}{\Gamma(2s + 6)\Gamma(s + 4)\Gamma(s + 3)\Gamma(s + 2)} \qquad (24)$$

for $\mathrm{Re}s > -3/2$. To see this, note that one may deduce directly from (19) that $2^{-6s}\langle|\hat{Z}(U_{[1,0]}, 0)|^s\rangle_{G_2}$ is bounded when $\mathrm{Re}s > 0$, and from Stirling's formula that

$$2^{-6s}\frac{\Gamma(3s+7)\Gamma(2s+3)}{\Gamma(2s+6)\Gamma(s+4)\Gamma(s+3)\Gamma(s+2)} \tag{25}$$

is also bounded in the same half-plane. The function

$$2^{-6s}\left(\langle|\hat{Z}(U_{[1,0]}, 0)|^s\rangle_{G_2} - \frac{\Gamma(3s+7)\Gamma(2s+3)}{\Gamma(2s+6)\Gamma(s+4)\Gamma(s+3)\Gamma(s+2)}\right) \tag{26}$$

is therefore regular and bounded in $\mathrm{Re}s > 0$, and vanishes when s is a non-negative integer. Carlson's theorem therefore implies (24).

Now that (24) has been established, we may immediately write down expressions for the probability density functions associated with the value distributions of $\log|\hat{Z}(U_{[1,0]}, 0)|$,

$$P_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma(3\mathrm{i}y+7)\Gamma(2\mathrm{i}y+3)}{\Gamma(2\mathrm{i}y+6)\Gamma(\mathrm{i}y+4)\Gamma(\mathrm{i}y+3)\Gamma(\mathrm{i}y+2)}\,\mathrm{e}^{-\mathrm{i}yx}\,\mathrm{d}y \tag{27}$$

and $|\hat{Z}(U_{[1,0]}, 0)|$,

$$P_2(x) = \frac{1}{2\pi\mathrm{i}x}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\Gamma(3s+7)\Gamma(2s+3)}{\Gamma(2s+6)\Gamma(s+4)\Gamma(s+3)\Gamma(s+2)}x^{-s}\,\mathrm{d}s \tag{28}$$

for any $c > 0$. One can easily deduce asymptotic properties of the probability density functions from these integrals (see, for example, [16, 17]).

We finally note that $\hat{Z}$ is real and positive at $\theta = 0$ and so its phase there is zero.

3.2. The 14-dimensional representation

In this case the determinant Z has a double zero at $\theta = 0$ (corresponding to the twice repeated weight 0, see (10)). Thus we define

$$\hat{Z}(U_{[0,1]}, \theta) = (1 - \mathrm{e}^{-\mathrm{i}\theta})^{-2}\det(1 - U_{[0,1]}\,\mathrm{e}^{-\mathrm{i}\theta}). \tag{29}$$

Similar calculations to the previous case show that for s a positive integer or zero, $\langle|\hat{Z}(U_{[0,1]}, 0)|^s\rangle_{G_2}$ is given by the constant term in

$$\frac{1}{12}\prod_{\alpha\in R}(1 - t^{\alpha})^{k_{\alpha}} \tag{30}$$

where in this case,

$$k_S = k_L = s + 1. \tag{31}$$

Thus for s a non-negative integer the Macdonald identity quoted above implies that

$$\langle|\hat{Z}(U_{[0,1]}, 0)|^s\rangle_{G_2} = \frac{(6s+6)!(2s+2)!}{12(5s+5)!((s+1)!)^3}. \tag{32}$$

Once again, Carlson's theorem may be applied (in this case after multiplication by 2^{-12s}) to show that

$$\langle|\hat{Z}(U_{[0,1]}, 0)|^s\rangle_{G_2} = \frac{\Gamma(6s+7)\Gamma(2s+3)}{12\Gamma(5s+6)[\Gamma(s+2)]^3} \tag{33}$$

for $\mathrm{Re}s > -7/6$. This can then be used to write down expressions for the probability density functions associated with the value distributions of $\log|\hat{Z}(U_{[0,1]}, 0)|$ and $|\hat{Z}(U_{[0,1]}, 0)|$, as in the previous section.

Also, as in the case of the representation [1, 0], the phase of $\hat{Z}$ is zero.

3.3. Other representations and $\theta \neq 0$

For other representations of G_2 and for $\theta \neq 0$, the integrand is not of the form

$$\prod_{\alpha \in R} (1 - t^\alpha)^{k_\alpha} \tag{34}$$

and, as far as we are aware, closed form expressions for these integrals are not known.

4. Value distribution problems over function fields

Our purpose now is to outline the analogy between number fields and function fields over a finite field in the context of the value distribution of zeta- and *L*-functions for these cases and the predicted behaviour in terms of RMT.

It was proved by Selberg that the logarithm of the Riemann zeta-function on the critical line has a Gaussian value distribution [24], and the same is true for all *L*-functions [26] under suitable assumptions. Selberg also investigated the '*q*-analogue' of these results for the value distribution of the family of Dirichlet *L*-functions [25] at a point on the critical line and there too obtained a Gaussian value distribution. Precisely, for *q* prime we have $q - 2$ primitive characters χ modulo *q*, and for fixed *t* consider the $q - 2$ numbers

$$\frac{\arg L\left(\frac{1}{2} + it, \chi\right)}{\sqrt{\frac{1}{2} \log \log q}}$$

(χ varies over all primitive/nonprincipal characters modulo *q*). Then as $q \to \infty$, these numbers are distributed as a standard Gaussian.

Our purpose in this section is twofold: first it is to point out that there are corresponding results for various families of *L*-functions over function fields, with the role of the Gaussian being replaced by various distributions from RMT; and second to construct an example for which the appropriate random-matrix distribution is the G_2 result calculated above.

Likewise, there are similar results for the *moments* of the *L*-functions, which in the number field setting are mostly conjectural [16, 17, 4], but in the function field setting can sometimes be proved.

4.1. Zeta-functions

Let *k* be a finite field of cardinality *q* and X/k a (smooth, geometrically connected, proper) curve defined over *k*. The zeta-function of X/k is given by the series

$$Z(X; T) = \exp\left(\sum_{n=1}^{\infty} N_n \frac{T^n}{n}\right) \tag{35}$$

where $N_n = \#X(k_n)$ is the number of points of *X* over the field k_n, the extension of *k* of degree *n*. The series is absolutely convergent for $|T| < 1$.

Trivial example: take $X = \mathbf{P}^1$, the projective line. The number of points of $\mathbf{P}^1$ over the finite field k_n is $\#k_n + 1 = q^n + 1$ and so

$$Z(\mathbf{P}^1; T) = \frac{1}{(1 - qT)(1 - T)}. \tag{36}$$

This zeta-function has an Euler product

$$Z(X; T) = \prod_p (1 - T^{\deg p})^{-1} \qquad |T| < 1 \tag{37}$$

where p runs over all closed points of X. (In the example of $\mathbf{P}^1$, the closed points p correspond to irreducible monic polynomials $p(x) \in k[x]$ with the addition of the 'point at infinity'.)

It turns out that $Z(X; T)$ is a *rational* function of T, of the form

$$Z(X; T) = \frac{P(X; T)}{(1 - T)(1 - qT)} \tag{38}$$

with $P(X; T) \in 1 + T\mathbf{Z}[T]$ a monic integer polynomial of degree $2g$, g being the genus of the curve X, which we can write as $P(X; T) = \prod_{j=1}^{2g}(1 - \alpha_j T)$. The inverse roots α_j are thus algebraic integers. Further, there is a functional equation $T \mapsto q/T$:

$$Z\left(X; \frac{1}{qT}\right) = q^{1-g} T^{2-2g} Z(X; T). \tag{39}$$

If we set $T = q^{-s}$ then the functional equation translates into $s \mapsto 1 - s$.

The 'Riemann hypothesis for curves over a finite field' (proved in the general case by Weil) is that all the inverse roots α_j of $P(X; T)$ have absolute value $\sqrt{q}$, that is as a function of the variable s all zeros are on the line $\operatorname{Re} s = 1/2$.

What is especially important for our purpose is that the polynomial $P(X; T)$ is the characteristic polynomial of a matrix: there is a unique conjugacy class $\Theta_X \in USp(2g)$ in the unitary symplectic group such that $P(X; T) = \det(I - q^{1/2} T \Theta_X)$.

4.2. Families of curves

Now consider a 'family' of curves X/k. In order to study the behaviour of $P(X; T)$ as X varies, it suffices to understand the distribution of the conjugacy classes Θ_X. In several cases, it is known that as $q \to \infty$ these become equidistributed in $USp(2g)$ (with respect to Haar measure).

For instance, this is the case for the family $\mathcal{M}_g$ of *all* k-isomorphism classes of (smooth, geometrically connected, proper) curves of given genus g [15, theorem 10.7.15].

This allows one to compute arithmetic quantities such as the *moments* of $P(X; T)$ as X varies in $\mathcal{M}_g(k)$ by using the corresponding (non-arithmetic) computation in random matrix theory for $USp(2g)$. Thus one finds that for T fixed, say $q^{-1}T = 1$, one has

$$\lim_{q \to \infty} \frac{1}{\#\mathcal{M}_g(k)} \sum_{X \in \mathcal{M}_g(k)} P(X, q^{1/2})^s = \int_{USp(2g)} \det(I - A)^s \, \mathrm{d}_{\mathrm{Haar}}(A). \tag{40}$$

The moments of the characteristic polynomial in $USp(2g)$ were computed in [17] and are given by

$$\int_{USp(2g)} \det(I - A)^s \, \mathrm{d}_{\mathrm{Haar}}(A) = 2^{2gs} \prod_{j=1}^{g} \frac{\Gamma(1 + g + j)\Gamma(1/2 + s + j)}{\Gamma(1/2 + j)\Gamma(1 + s + g + j)}. \tag{41}$$

The probability density functions for the value distributions associated with the polynomial and its logarithm may then be written as integrals, as in (27) and (28) [17]. In the case of the logarithm of the characteristic polynomial, the limit distribution when $g \to \infty$ is a Gaussian.

4.3. L-functions attached to exponential sums

We consider one-variable exponential sums constructed as follows: let k be a finite field with q elements as above, $f(x)$ and $h(x) \in k(x)$ be rational functions, ψ a nontrivial additive character of k (e.g., for $k = \mathbf{Z}/p\mathbf{Z}$ take $\psi(x) = \exp(2\pi \mathrm{i} ax/p), 0 \neq a \in \mathbf{Z}/p\mathbf{Z}$) and χ a multiplicative character of $k^{\times}$. Set

$$S(\psi, \chi; f, h; q) = \sum_{x} \psi(f(x))\chi(h(x)) \tag{42}$$

the sum running over all $x \in k$ which are not poles of f, h and such that $h(x) \neq 0$. For the finite extension k_n of degree n of k, we get nontrivial characters $\psi_n = \psi \circ \mathrm{Tr}_{k_n/k}$ and $\chi_n = \psi \circ \mathrm{N}_{k_n/k}$ by composing with the trace and norm maps. Correspondingly we get exponential sums for k_n,

$$S_n(\chi, \psi; f, h) := S(\chi_n, \psi_n; f, h; q^n). \tag{43}$$

The *L*-function is defined as

$$L(S, T) = \exp\left(\sum_{n=1}^{\infty} S_n(\chi, \psi; f, h)\frac{T^n}{n}\right). \tag{44}$$

These have an Euler product decomposition and are rational functions of T.

In many cases of interest to us, it turns out that $L(T)$ is in fact a polynomial of the form $\det(I - q^{1/2}T\Theta_S)$ with Θ_S a unitary matrix.

4.4. Gauss sums

Given a nontrivial additive character ψ of k and a nontrivial multiplicative character χ of $k^\times$, one defines the Gauss sum $g(\chi, \psi)$ by

$$g(\chi, \psi) = \sum_{x \neq 0} \chi(x)\psi(x). \tag{45}$$

Correspondingly we get Gauss sums for k_n

$$g_n(\chi, \psi) := g(\chi_n, \psi_n). \tag{46}$$

To compute the *L*-function

$$L(g(\chi, \psi), T) = \exp\left(\sum_{n=1}^{\infty} g_n(\chi, \psi)\frac{T^n}{n}\right) \tag{47}$$

one can use the Hasse–Davenport relations:

$$-g_n(\chi, \psi) = (-g(\chi, \psi))^n. \tag{48}$$

These give

$$L(g(\chi, \psi), T) = 1 + Tg(\chi, \psi). \tag{49}$$

As is well known, $|g(\chi, \psi)| = \sqrt{q}$. Thus we may write $g(\chi, \psi) = \sqrt{q}\, e^{i\theta_\chi}$ (we omit the dependence on ψ which is of a trivial nature). Setting $s = 1/2 + i\theta/\log q$, $T = q^{-s} = q^{-1/2}\, e^{-i\theta}$, we find

$$L(g(\chi, \psi), T) = 1 + e^{i(\theta_\chi - \theta)}. \tag{50}$$

The $q - 2$ angles $\{\theta_\chi : \chi \neq \chi_0\}$ are uniformly distributed in $[0, 2\pi)$ as $q \to \infty$. This is easy to see from Deligne's estimate on hyper-Kloosterman sums (see [12, section 1.3.3]). Thus the moments of $L(g(\chi, \psi), T)$ and of its logarithm, averaged over χ and taken as $q \to \infty$, are the same as those for the function $1 + e^{i\theta}$.

4.5. Kloosterman sums (i)

These are the sums

$$\mathrm{Kl}(a, p) = \sum_{x_1 x_2 = a \bmod p} \exp\frac{2\pi i}{p}(x_1 + x_2). \tag{51}$$

More generally for a finite field k with q elements, take a nontrivial additive character ψ and $a \neq 0$, and set

$$\text{Kl}(a, q) = \sum_{x_1 x_2 = a} \psi(x_1 + x_2). \tag{52}$$

This sum is real (replace $x \mapsto -x$), and as Weil proved satisfies

$$|\text{Kl}(a, q)| \leqslant 2\sqrt{q}. \tag{53}$$

The associated L-function is a polynomial of degree 2,

$$L(\text{Kl}(a, q), T) = 1 + \text{Kl}(a, q)T + qT^2. \tag{54}$$

It is of the form $\det(I - q^{1/2}T\Theta_a)$ with $\Theta_a \in SU(2)$. It was shown by Katz [13] that as $q \to \infty$, the $q - 1$ conjugacy classes $\{\Theta_a : a \in k^{\times}\}$ become equidistributed in $SU(2)$ with respect to Haar measure. This implies that we can compute the value distribution of L and $\log L$ via RMT on $SU(2)$.

4.6. Kloosterman sums (ii)

We next look at an example of exponential sums in several variables: hyper-Kloosterman sums are n-variable sums ($n \geqslant 2$) generalizing the previous example, given by

$$\text{Kl}_n(a, q) = \sum_{x_1 x_2 \ldots x_n = a} \psi(x_1 + x_2 + \cdots + x_n). \tag{55}$$

Replacing ψ by $\psi \circ \text{Tr}_{k_m/k}$ gives the sum $\text{Kl}_n(a, q^m)$.

The associated L-function is defined as

$$L_a(T) := \exp\left((-1)^n \sum_{m=1}^{\infty} \text{Kl}_n(a, q^m) \frac{T^m}{m} \right). \tag{56}$$

This L-function is a polynomial of degree n. It was proved by Katz [13] that it can be written as $\det(I - q^{(n-1)/2}T\Theta_n(a, q))$, with $\Theta_n(a, q) \in K_n$, where K_n is the compact group $USp(2n)$, n even, $SU(n)$, n, q odd, $SO(n)$, q even, $n \neq 7$ odd, and G_2, q even, $n = 7$. Moreover, as a varies through all nonzero elements of k, the $q - 1$ conjugacy classes $\Theta_n(a, q)$ of K_n become equidistributed there as $q \to \infty$ while keeping the type of K_n fixed. For instance, taking $n = 7$ and $q = 2^r, r \to \infty$ gives $2^r - 1$ conjugacy classes $\{\Theta_7(a, 2^r) : 0 \neq a \in \mathbf{F}_{2^r}\}$ which become equidistributed in G_2 as $r \to \infty$.

4.7. An exponential sum associated with G_2

Let p be a prime, $p \geqslant 17$, $k = \mathbf{Z}/p\mathbf{Z}$, $\chi_{(2)}$ the unique quadratic character (Legendre symbol) of $k^{\times}$, and ψ a nontrivial additive character of k, that is $\psi(x) = e^{2\pi iax/p}$ for some $a \in k^{\times}$. Consider for $t \in k^{\times}$ the exponential sum

$$\mathcal{KT}(t) = \sum_{x \in k^{\times}} \chi_{(2)}(x)\psi(x^7 + tx). \tag{57}$$

These sums were studied by Katz and the results below are due to him [14].

Note that $\overline{\mathcal{KT}(t)} = \chi_{(2)}(-1)\mathcal{KT}(t)$ and so $\mathcal{KT}(t)$ is real if $\chi_{(2)}(-1) = 1$, that is if $p = 1 \bmod 4$, and imaginary if $\chi_{(2)}(-1) = -1$, i.e. if $p = 3 \bmod 4$.

In view of the transformation properties under complex conjugation, we divide the exponential sum $\mathcal{KT}(t)$ by the quadratic Gauss sum $g(\chi_{(2)})$ to get a *real* number. Furthermore, there is a (unique) choice of sign $\epsilon_p = \pm 1$ so that[3]

$$\mathcal{KT}'(t) = \epsilon_p \frac{\mathcal{KT}(t)}{g(\chi_{(2)})} \tag{58}$$

is minus the trace of a matrix $\Theta_t \in SO(7)$: $\mathcal{KT}(t) = -\mathrm{Tr}\Theta_t$. Moreover, this matrix turns out to lie in G_2.

The associated *L*-function is a polynomial of degree 7, which is a characteristic polynomial of the element Θ_t of G_2,

$$L(\mathcal{KT}'(t), T) = \det(I - \Theta_t T). \tag{59}$$

As t varies in $(\mathbf{Z}/pZ)^\times$, these $p - 1$ conjugacy classes Θ_t become equidistributed in G_2 as $p \to \infty$. Thus the value distribution of $L(\mathcal{KT}'(t), T)$ at fixed T is computed by RMT for G_2.

5. Other Lie groups

The random matrix calculations reported here for G_2 generalize straightforwardly to the other exceptional Lie groups. In each case one has a Weyl integration formula, which allows the moments of the characteristic polynomials associated with representations of the group to be written as integrals over the Cartan subgroup, and a Macdonald identity, which enables the integrals to be evaluated as ratios of Γ-functions. This prompts the question as to whether families of finite-field *L*-functions can be constructed whose value distributions are given by each of the other exceptional Lie groups.

As a final remark, we note that in [18] Macdonald gives constant term formulae for affine root systems (which are related to Kac-Moody algebras). This suggests the intriguing possibility of extending the ideas described in this paper to families of random matrices arising from the representations of the associated infinite-dimensional groups.

Acknowledgments

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